Small ball probabilities, maximum density and rearrangements

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Abstract

We prove that the probability that a sum of independent random variables in $\mathbb{R}^d$ with bounded densities lies in a ball is maximized by taking uniform distributions on balls. This in turn generalizes a result by Rogozin on the maximum density of such sums on the line.

Let μ be the Lebesgue measure on $\mathbb{R}^d$, and let X be a random vector in $\mathbb{R}^d$. If X has a density p, we define

$$M(X) = \operatorname{ess\,sup} p := \sup\{\epsilon : \mu(\{t : p(t) > \epsilon\}) > 0\}.$$

For random variables with distributions that are not absolutely continuous with respect to μ measure we set $M(X) := \infty$. We note that ess sup is invariant under changes to p on sets of measure 0. Hence we will take our density functions to be equivalence classes up to alterations on sets of measure 0; that is, they are defined as elements of L_∞.

The aim of this paper is to provide best possible upper bounds for the maximum density and small ball probabilities of sums of random vectors.

Our starting point is a result by Rogozin, who showed that in the case $d = 1$ the worst case is provided by uniform distributions over intervals. To be more precise, it was proved in [4] that for independent real random variables $X_1, \ldots, X_n$ with $M(X_i) \le M_i$ we have

$$M(X_1 + \cdots + X_n) \le M(U_1 + \cdots + U_n),$$

where U_k are independent and uniformly distributed in $[-\frac{1}{2M_i}, \frac{1}{2M_i}]$.

We extend Rogozin's inequality to all dimensions. In fact, we prove a more general statement for small ball probabilities that immediately implies a generalisation of Rogozin's result.

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Theorem 1. *Let $X_1, \ldots, X_n$ be independent random vectors in $\mathbb{R}^d$ with $M(X_i) \leq K_i$. Consider a collection of independent random vectors $U_1, \ldots, U_n$ with densities equal to K_i on a ball around the origin and 0 elsewhere. Then for every measurable set S we have*

$$\mathbb{P}\left(X_1 + \cdots + X_n \in S\right) \leq \mathbb{P}\left(U_1 + \cdots + U_n \in B\right), \tag{1}$$

where B is the centered ball such that $\mu(B) = \mu(S)$.

Corollary 1. *Under the same conditions as above we also have that*

$$M(X_1 + \cdots + X_n) \leq M(U_1 + \cdots + U_n).$$

Proof. Note that for any variable X with density p

$$M(X) = \lim_{\epsilon \to 0} \sup_{\mu(S)=\epsilon} \epsilon^{-1} \int_S p d\mu,$$

and from Theorem 1 for every fixed ϵ the right hand side is not decreased by taking the variables U_i in place of X_i. Hence the corollary holds. □

Even for $d = 1$ our approach to Theorem 1 is quite different than that of Rogozin, who used discretization arguments together with an idea of Erdős to relate small ball probabilities to Sperner's theorem in finite set combinatorics. We avoid these subtleties by using a rearrangement inequality proved by Brascamp, Lieb and Luttinger.

Before stating this result, we define the spherically symmetric decreasing rearrangement. Given a non-negative function $f : \mathbb{R}^d \mapsto \mathbb{R}$ we first set $M_y^f = \{t : f(t) \geq y\}$. Suppose we are given an f such that $M_a^f < \infty$ for some $a \in \mathbb{R}$. We define $\tilde{f}$ to be a function such that:

$$\begin{aligned} &1)\ \tilde{f}(x) = \tilde{f}(y), \text{ for} |x|_2 = |y|_2; \\ &2)\ f(x) \leq f(y) \text{ for } x \leq y; \\ &3)\ M_y^{\tilde{f}} = M_y^f. \end{aligned}$$

The function $\tilde{f}$ is known as the spherically symmetric decreasing rearrangement of f. For existence, uniqueness and other properties of $\tilde{f}$ we refer the reader to [2] and [3].

Having introduced the relevant symmetrization we can state the aforementioned rearrangement result.

Theorem 2. *Let f_j, $1 \leq j \leq k$ be non-negative measurable functions on $\mathbb{R}^d$ and let $a_{j,m}$, $1 \leq j \leq k$,$1 \leq m \leq n$, be real numbers. Then*

$$\int_{\mathbb{R}^{nd}} \prod_{j=1}^{k} \left(f_j \left(\sum_{m=1}^{n} a_{j,m} x_m \right) \right) d^{nd} \leq \int_{\mathbb{R}^{nd}} \prod_{j=1}^{k} \left(\tilde{f}_j \left(\sum_{m=1}^{n} a_{j,m} x_m \right) \right) d^{nd}$$

A direct consequence of the latter result is the following.

Theorem 3. *Let $X_1, \dots, X_n$ be independent random variables with given density functions p_i. Consider another collection of independent random variables $X_1', \dots, X_n'$ with density functions $\tilde{p}_i$. Then for every measurable set S we have*

$$\mathbb{P}\left(X_1 + \cdots + X_n \in S\right) \le \mathbb{P}\left(X_1' + \cdots + X_n' \in B\right), \tag{2}$$

where B is the centered ball such that $\mu(B) = \mu(S)$.

Proof. We have that

$$\mathbb{P}\left(\sum X_i \in S\right) = \int_{x_1,\dots x_n} \prod_{i=1}^{n} p_i(x_i) \mathbb{1}_S \left(\sum_i x_i\right) d^n\mu.$$

Now apply Theorem 2 with the f_i taken to be $\{p_1, \dots, p_n, \mathbb{1}_S\}$ and the $a_{j,m} = 1$ when $j = m$ or $j = n+1$ and $a_{j,m} = 0$ otherwise. We note that $\widetilde{\mathbb{1}_S} = \mathbb{1}_B$ and that $\tilde{p}_i$ are the densities of X_i', completing the proof. ☐

To obtain Theorem 1 we will first characterize the extreme points of the set of measures with bounded densities.

Lemma 1. *Let $\mathcal{S}_K$ be the set of probability measures in $\mathbb{R}^d$ that have essential suprema bounded by $K > 0$. The extreme points of $\mathcal{S}_K$ are measures having densities $p(t) = K\mathbb{I}_S(t)$ for some set S with $\mu(S) = 1/K$.*

Proof. Firstly, we note that all measures having densities $p = K\mathbb{I}_S$ are extremal. Suppose not. Then $p = \alpha p_1 + (1-\alpha)p_2$, where $\alpha \in (0,1)$ and p_1, p_2 are not equal to p. But then p_1 and p_2 differ from p on a set of positive measure, and so $\max(p_1, p_2) > K$ on some set of positive measure. Hence one of p_1, p_2 must exceed K on a set of positive measure, so is outside of $\mathcal{S}_K$.

Suppose that the density of a measure is not one of these extremal examples. Consider the sets

$$A_y = \{t : p(t) \ge y\}.$$

Now, there is some $y \in (0, K)$ such that $\mu(A_y) > 0$, as otherwise $p(t) = K$ almost everywhere on its support, and so p would be one of our extremal examples. We fix any such y, and define $X = \sup(p) \backslash A_y$. Furthermore, we partition X into two disjoint sets X_1, X_2 such that $\int_{X_1} p d\mu = \int_{X_2} p d\mu$.

We fix $\delta \in (0, K/y - 1) \cap (0, 1)$, and construct two densities p_1, p_2 as follows:

$$p_i(t) = \begin{cases} p(t) & t \in A_y \\ (1-\delta)p(t) & t \in X_i \\ (1+\delta)p(t) & t \in X_{1-i} \end{cases}$$

First, we observe that $p = \frac{1}{2}(p_1 + p_2)$. Furthermore, each of p_1, p_2 are equal to p on A_y, and are bounded pointwise on X by:

$$(1+\delta)\sup_X p \leq (1+\delta)y \leq K.$$

Hence the essential suprema of p_1, p_2 are bounded by K, and so $p_1, p_2 \in \mathcal{S}_K$ as required. $\square$

We now prove Theorem 1:

Proof. We first observe that Equation 1 can be written as a multilinear integral over the densities of X_i and the indicator function of S. As a corollary, it is maximized when each p_i is an extremal member of $\mathcal{S}_{K_i}$.

Hence from Lemma 1 each density p_i is proportional to the indicator function of a set of measure K_i^{-1}. From Theorem 3, we have that to maximize this expression we may replace each of the densities p_i by $\tilde{p}_i$ and replace S by a ball B of the same volume.

We now observe that if p_i is proportional to an indicator function, then $\tilde{p}_i$ is proportional to the indicator function of a ball centered on the origin, which completes the theorem. $\square$

References

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